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AN ATTRITION MODEL
FOR PENETRATION PROCESSES

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IDA INSTITUTE FOR DEFENSE ANALYSES
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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This paper presents a new model of combat attrition between attackers and defenders. The model represents cases where the defenders are protecting a passive target and the attackers are attempting to penetrate the defenders to attack the target. A complete set of assumptions and rigorous derivations of the relevant attrition processes are given. Interpretations of the assumptions, extensions,

20. continued

and computational aspects are considered. A taxonomy relating this
attrition model with several other attrition equations is also given.

PREFACE

The purpose of this paper is to describe what appears to
be a new probabilistic model of combat attrition. While related
to the binomial attrition model discussed in KARR (1974), this
model is different in two important respects. First, the mathe-
matical assumptions differ and this, we believe, is the most
significant way of comparing mathematical models. Second, this
new model allows simple computation of the probability distri-
butions of relevant random variables, rather than just the
expectations. Section 1 of this paper concerns the mathematical
assumptions of our attrition process and the characterizations,
derived from the assumptions, of various stochastic processes of
interest. In Section 2 we isolate consideration of physical
interpretations of our mathematical assumptions, and we discuss
combat situations which might in some sense satisfy the assump-
tions. Sections 3 and 4 deal, respectively, with generalizations
and computational aspects of the basic model. This research was
motivated by a review by ANDERSON (1972) of attrition processes
used in several air-to-air models. In the course of this research
we became aware that some similar but less complete and less
rigorous results are given in WHITAKER (1970).

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TABLE

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1. MATHEMATICAL ASSUMPTIONS AND RESULTS

Our model describes a bilateral combat attrition process involving a set of defenders and a set of penetrators (or attackers). The assumptions we give here attempt to be as free as possible of restrictive physical interpretation, but obviously cannot be entirely so. Roughly speaking, one should envision the defenders as protecting some target that the penetrators wish to attack.

Here are our assumptions:

1. Penetrators attempt to penetrate the defenses and reach the target successively, one after another.

2. A penetrator attempting penetration of the defenses is detected by each defender present with probability d, independent of detections by any other defenders and of the past history of the process.

3. If the penetrator is detected by one or more defenders then exactly one defender is assigned to engage the penetrator in a one-on-one duel.

4. An engagement between a penetrator and a defender ends in one of the following outcomes with the respective probabilities shown, independent of the past history of the process.

Outcome of Engagement	Probability
Destruction of both	p_1
Destruction of penetrator only	p_2
Destruction of defender only	p_3
Destruction of neither	p_4

Clearly, $0 \le p_i \le 1$ $(i=1,2,3,4)$ and $p_1 + p_2 + p_3 + p_4 = 1$.

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5. A defender that survives a duel is unable to return to the set of active defenders; a penetrator that survives a duel must turn back without attempting to attack the target.

All action is assumed to occur within some fixed period of time.

We now define some quantities of interest in this attrition model, whose probability distributions and expectations we shall then proceed to compute.

Let D_0 be the initial number of active defenders (i.e., the number of defenders before any penetrators have attempted penetration). A feature of this model is that D_0 can be a random variable. In general, the random variable D_0 could assume the value of any nonnegative integer. But both for computational reasons and for realistic modeling (there never is an infinite number of defenders) we assume that there is an upper bound, M, on the number of defenders. Thus, we assume that D_0 is concentrated on the integers $0,1,\dots,M$. Let

D_k = number of active defenders remaining after k penetrators have attempted penetrations;

B_k = number of engagements that occur involving one of the first k penetrators;

X_k = number of defenders destroyed by the first k penetrators;

Y_k = number of the first k penetrators destroyed;

U_k = number of defenders engaged, but not destroyed, by the first k penetrators; and

V_k = number of the first k penetrators which are engaged by defenders, but not destroyed.

Obviously,

$$B_k + D_k = D_0 ;$$

(1)
$$X_k + U_k = B_k ;$$

$$Y_k + V_k = B_k ,$$

for each k.

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The stochastic process $D = (D_k)_{k \geq 0}$ describing the evolution of the set of active defenders is the key to the analysis of this attrition model and may be characterized in the following manner:

(2) THEOREM. The stochastic process D is a Markov process with transition matrix P given by

$$P(0,0) = 1 \ ,$$

while for $i \geq 1$,

$$P(i,j) = \begin{cases} 1 - (1 - d)^i \ , & \text{if } j = i - 1 \\ (1 - d)^i \ , & \text{if } j = i. \end{cases}$$

PROOF. Suppose at some point there remain i active defenders; since future evolution of the process is independent of the past once this number i is known, D has the Markov property. The next arriving penetrator remains undetected by all defenders with probability $(1 - d)^i$, by Assumption 2, in which case the number of active defenders remains at i. Otherwise, with the complementary probability $1 - (1 - d)^i$, the penetrator is detected and the number of active defenders decreases to $i - 1$. ▯

The transition matrix P has the following simple form:

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ . \\ . \\ . \\ M \end{array} \begin{bmatrix} 0 & 1 & 2 & \ldots & M-1 & M \\ 1 & 0 & 0 & & & \\ d & (1-d) & 0 & & & \\ 0 & 1-(1-d)^2 & (1-d)^2 & & & \\ & & . & . & & \\ & & & . & . & \\ & & & & . & . \\ & & & & 1-(1-d)^M & (1-d)^M \end{bmatrix} ,$$

which facilitates efficient calculation on a computer of the
powers of P used in the following further computations. In
Section 4 we give a closed form computation of the powers of P.

The engagement process $B = (B_k)_{k \geq 0}$ is somewhat harder to
describe. Because of the presence of the random variable D_0 in
the representation (1), this process is not a Markov process.
But we can state two descriptions: the first is interesting in
an abstract sense and the second is of practical computational
value.

(3) THEOREM. Conditioned on the random variable D_0, B is a
Markov process with transition matrix $\hat{P}$ given by

$$\hat{P}(i,j) = P(D_0 - i, D_0 - j) .$$

We omit the straightforward proof; but note the proof of
the following result.

(4) THEOREM. For each k and for $j \leq k$ we have

$$P\{B_k = j\} = \sum_{\ell=0}^{M} P\{D_0 = \ell\} P^k(\ell, \ell-j) .$$

PROOF. Using properties of conditional probabilities (this
argument essentially gives Theorem (3) as well) we have

$$P\{B_k = j\} = E[P\{B_k = j | D_0\}]$$
$$= E[P\{D_0 - D_k = j | D_0\}]$$
$$= E[P\{D_k = D_0 - j | D_0\}]$$
$$= E[P^k(D_0, D_0 - j)]$$

(where P^k is the kth power of the transition matrix P)

$$= \sum_{\ell=0}^{M} P\{D_0 = \ell\} P^k(\ell, \ell-j) . \qquad \square$$

An additional advantage of this new attrition process is the ease with which it can accept random variables as inputs, something many other models are not able to do. See Table 1 for details.

We next compute the probability distributions for the attrition processes $X = (X_k)_{k \geq 0}$ and $Y = (Y_k)_{k \geq 0}$.

(5) THEOREM. For $\ell \leq k$ we have

$$P\{X_k = \ell\} = \sum_{m=\ell}^{k} \binom{m}{\ell} (p_1 + p_3)^{\ell} (p_2 + p_4)^{m-\ell} P\{B_k = m\}$$

and

$$P\{Y_k = \ell\} = \sum_{n=\ell}^{k} \binom{n}{\ell} (p_1 + p_2)^{\ell} (p_3 + p_4)^{n-\ell} P\{B_k = n\} .$$

PROOF. Given that $B_k = m$, the number of defenders destroyed is, according to Assumption 4, binomially distributed with parameters $(m, p_1 + p_3)$, so the first expression follows. The second is entirely analogous. ∏

By exactly the same methods, we obtain probability distributions for the processes U and V, included here for the sake of completeness.

(6) THEOREM. Provided $\ell \leq k$

$$P\{U_k = \ell\} = \sum_{m=\ell}^{k} \binom{m}{\ell} (p_2 + p_4)^{\ell} (p_1 + p_3)^{m-\ell} P\{B_k = m\}$$

and

$$P\{V_k = \ell\} = \sum_{n=\ell}^{k} \binom{n}{\ell} (p_3 + p_4)^{\ell} (p_1 + p_2)^{n-\ell} P\{B_k = n\} .$$

The proof of Theorem (6) is also omitted since it follows from (1) that

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$$P\{U_k = \ell \,|\, B_k = m\} = P\{X_k = m - \ell \,|\, B_k = m\}$$

and

$$P\{V_k = \ell \,|\, B_k = n\} = P\{Y_k = n - \ell \,|\, B_k = n\} \ ,$$

so that Theorem (6) is actually a corollary to Theorem (5).

There are other quantities of interest whose probability distributions one would like to obtain. Let

R_k = number of defenders surviving (but not necessarily active) after k attempted penetrations,

S_k = number of the first k penetrators which are <u>not</u> detected and engaged by defenders,

T_k = number of the first k penetrators surviving interactions (if any) with defenders.

We remind the reader that defenders involved in but surviving an interaction would be able to participate in a future battle (but not in the currently ongoing battle) and that the same applies to penetrators that are engaged but not destroyed (those denoted by T_k above). The unengaged penetrators (S_k above) can proceed to attack their target.

Clearly

$$R_k = D_0 - X_k = D_k + U_k$$

(7)
$$S_k = k - B_k$$

$$T_k = k - Y_k = S_k + V_k \ .$$

(8) THEOREM. For $\ell \geq M - k$, we have

$$P\{R_k = \ell\} = \sum_{i=\ell}^{M} \sum_{j=i-\ell}^{k} \binom{j}{i-\ell} (p_1 + p_3)^{i-\ell} (p_2 + p_4)^{j-i+\ell} p^k(i, i-j) P\{D_0 = i\}$$

$$P\{S_k = \ell\} = P\{B_k = k - \ell\}$$

$$P\{T_k = \ell\} = P\{Y_k = k - \ell\} \ .$$

6

We omit the proof, which interested readers can supply for themselves.

So far we have considered fixed numbers of penetrators. What happens if the number of penetrators attempting to penetrate the defense within the time period under consideration is a random variable A? Essentially all the necessary mathematics is done. For example, suppose we wish to compute the probability distribution of the number of defenders killed, which in this case is the random variable X_A. Then, assuming that A is independent of D_0 and the attrition process and that N is an upper bound on the number of penetrators,

$$
\begin{aligned}
P\{X_A = q\} &= \sum_{k=0}^{N} P\{X_A = q, A = k\} \\
&= \sum_{k=0}^{N} P\{X_k = q, A = k\} \\
&= \sum_{k=0}^{N} P\{X_k = q\}P\{A = k\}
\end{aligned}
$$

which is immediately calculable in terms of the probability distribution of A and quantities given in Theorem (5). Similar comments apply to the other stochastic processes we have defined.

In Table 1 below we summarize how all relevant probability distributions can be computed in terms of the probability distributions of the numbers of defenders and penetrators, the detection probability d, and the kill probabilities p_1, p_2, p_3, p_4. We also include the expectations of these random variables.

In some cases the identities (1) and (7) allow simplification of the expressions for expectations, and we have done so in Table 1. Indeed, all expectations depend only on $E[D_0]$, $E[A]$, and $E[D_A]$.

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Table 1. COMPUTATION OF PROBABILITY DISTRIBUTIONS

Given Data: $\mu(i) = P\{D_0=i\}$ (where $0 \leq i \leq M$ and D_0 is the initial number of defenders)

$\lambda(j) = P\{A=j\}$ (where $0 \leq j \leq N$ and A is the number of penetrators)

P_1, P_2, P_3, P_4 = engagement outcome probabilities (see Assumption 4)

d = probability of detection (see Assumption 2)

$$
P^j(i,k) = \begin{cases}
\displaystyle\prod_{\ell=k+1}^{i} (1-(1-d)^\ell) \sum_{r=k}^{i} \frac{\cdot\ (1-d)^{rj}}{\displaystyle\prod_{\substack{q=k \\ q \neq r}}^{i} ((1-d)^r - (1-d)^q)} & \begin{array}{l} k < i, \\ i-k \leq j \end{array} \\[30pt]
(1-d)^{ij} & k=i \\[10pt]
0 & \text{otherwise}
\end{cases}
$$

(see Theorem 16)

1. D_A = number of remaining active defenders after all attempted penetrations

$$P\{D_A=k\} = \sum_{i=0}^{M} \sum_{j=0}^{N} \mu(i)\lambda(j)P^j(i,k)$$

$$E[D_A] = \sum_{i=0}^{M} \sum_{j=0}^{N} \sum_{k=0}^{i} \mu(i)\lambda(j)kP^j(i,k)$$

2. B_A = number of one-on-one engagements

$$P\{B_A=k\} = \sum_{i=0}^{M} \sum_{j=0}^{N} \mu(i)\lambda(j)P^j(i,i-k)$$

$$E[B_A] = E[D_0] - E[D_A]$$

$$= \sum_{i=0}^{M} i\mu(i) - \sum_{i=0}^{M} \sum_{j=0}^{N} \sum_{k=0}^{i} \mu(i)\lambda(j)kP^j(i,k)$$

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Table 1 (continuted)

3. X_A = number of defenders destroyed

$$P\{X_A=k\} = \sum_{i=0}^{M} \sum_{j=0}^{N} \sum_{\ell=0}^{i} \mu(i)\lambda(j)P^j(i,i-\ell)\binom{\ell}{k}(p_1+p_3)^k(p_2+p_4)^{\ell-k}$$

$$E[X_A] = (p_1+p_3)E[B_A]$$

$$= (p_1+p_3)\left(\sum_{i=0}^{M} i\mu(i) - \sum_{i=0}^{M} \sum_{j=0}^{N} \sum_{k=0}^{i} \mu(i)\lambda(j)kP^j(i,k)\right)$$

4. Y_A = number of penetrators destroyed

$$P\{Y_A=k\} = \sum_{i=0}^{M} \sum_{j=0}^{N} \sum_{\ell=0}^{i} \mu(i)\lambda(j)P^j(i,i-\ell)\binom{\ell}{k}(p_1+p_2)^k(p_3+p_4)^{\ell-k}$$

$$E[Y_A] = (p_1+p_2)E[B_A]$$

$$= (p_1+p_2)\left(\sum_{i=0}^{M} i\mu(i) - \sum_{i=0}^{M} \sum_{j=0}^{N} \sum_{k=0}^{i} \mu(i)\lambda(j)kP^j(i,k)\right)$$

5. R_A = number of defenders surviving at the end of the time period under consideration

$$P\{R_A=k\} = \sum_{i=0}^{M} \sum_{j=0}^{N} \sum_{m=0}^{i} \mu(i)\lambda(j)P^j(i,i-m)\binom{m}{i-k}(p_2+p_4)^{m-i+k}(p_1+p_3)^{i-k}$$

$$E[R_A] = E[D_0] - E[X_A]$$

$$= (p_2+p_4)\left(\sum_{i=0}^{M} i\mu(i) + (p_1+p_3)\sum_{i=0}^{M} \sum_{j=0}^{N} \sum_{k=0}^{i} \mu(i)\lambda(j)kP^j(i,k)\right)$$

(concluded on next page)

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Table 1 (concluded)

6. S_A = number of penetrators able to attack the target

$$P\{S_A=k\} = \sum_{i=0}^{M} \sum_{j=0}^{N} \mu(i)\lambda(j)P^j(i,i-j+k)$$

$$E[S_A] = E[A] - E[B_A]$$

$$= \sum_{j=0}^{N} j\lambda(j) - \sum_{i=0}^{M} i\mu(i) + \sum_{i=0}^{M} \sum_{j=0}^{N} \sum_{k=0}^{i} \mu(i)\lambda(j)kP^j(i,k)$$

7. T_A = number of penetrators surviving interactions (if any) with defenders

$$P\{T_A=k\} = \sum_{i=0}^{M} \sum_{j=0}^{N} \sum_{m=0}^{i} \mu(i)\lambda(j)P^j(i,i-m)\binom{m}{j-k}(p_1+p_3)^{j-k}(p_2+p_4)^{m-j+k}$$

$$E[T_A] = E[A] - E[Y_A]$$

$$= \sum_{j=0}^{N} j\lambda(j) - (p_1+p_2)\sum_{i=0}^{M} i\mu(i)$$

$$+ (p_1+p_2)\sum_{i=0}^{M} \sum_{j=0}^{N} \sum_{k=0}^{i} \mu(i)\lambda(j)kP^j(i,k)$$

2. DISCUSSION AND INTERPRETATION OF THE ASSUMPTIONS

In this section we will do three things. First, briefly
list some types of combat that could be considered as penetra-
tion processes. Second, discuss some alternative models of
barrier penetration processes so that they can be compared to
the new process presented here. Third, describe some physical
situations in which the assumptions of our attrition model,
as presented in Section 1, are satisfied to some extent.

a. Penetration Processes

As described by KARR (1975), a number of conventional
combat situations can be classified into three categories.
First, each side could try to kill the other, or they both could
try to control the same territory. Second, one side could try
to maintain a barrier through which the other side attempts to
penetrate in order to attack targets beyond the barrier. And
third, one side could attempt to destroy passive targets on
the other side. The attrition model given here is not appro-
priate for the first category of combat; it should be considered
as a candidate for describing the second category; and it might
be considered for describing the third category--if one con-
siders the attackers as a moving barrier that passes over (or
under) the passive targets.

Some examples of barrier penetration processes that can
occur in conventional combat are as follows: (1) interceptor
aircraft through a screen of escort aircraft; (2) attack air-
craft through interceptors; (3) attack aircraft through SAMs
and AAA; (4) submarines through a barrier consisting of enemy

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submarines; (5) submarines through search aircraft; (6) sub-
marines through naval ships escorting convoys; (7) soldiers
through enemy lines; and (8) anything through a minefield.

b. Comparison with Alternative Assumptions

Depending on the particular details of the above examples,
the attrition model presented here may or may not be an appro-
priate description. Assumptions 1, 2, and 3 form a basis for
comparing our model with other attrition models that could be
used to describe barrier penetration. Some alternative forms
of Assumption 1 are as follows: (a) SIMULTANEOUS--the penetra-
tors arrive at the barrier simultaneously and are simultaneously
vulnerable to all defenders; (b) PENETRATOR SEQUENTIAL--assump-
tion 1 as stated in Section 1; (c) DEFENDER SEQUENTIAL--pene-
trators arrive at the barrier simultaneously, but the defenders
are one behind another in the barrier so that the penetrators
pass by the defenders sequentially; (d) DOUBLE SEQUENTIAL--pene-
trators arrive sequentially and pass by the defenders sequentially.

We will not attempt to construct plausible alternatives to
Assumptions 2 or 3 for the Defender Sequential and the Double
Sequential cases. We consider the following alternatives for
the Simultaneous and Penetrator Sequential cases: An alterna-
tive to Assumption 2 (which we will refer to below as Individual
Detection) is that there is one central detector, such as an
AWACS, and that the defenders can be coordinated and assigned to
engage any particular detected penetrator.[1] In this alternative,
d is defined as the probability that the central detector detects
a particular penetrator, so the probability that a particular
penetrator is detected is independent of the number of defenders.

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[1] This does not mean that we recommend only this alternative for modeling
attrition considering AWACS. A combination of the processes described
below might be more appropriate. For example, the AWACS could make an
initial detection and send a group of defenders to engage a group of
attackers, and the engagement between groups could be modeled by one of
the other alternative sets of assumptions.

We will refer to this alternative as Coordinated Central Detection. An alternative to Assumption 3 (one-on-one combat) is that two or more defenders can engage a penetrator (many-on-one combat). Graphically, these possibilities can be displayed as follows:

Encounters

	One-on-One	Many-on-One
SIMULTANEOUS (I.D.)	A	B
SIMULTANEOUS (C.C.D.)	C	D
PENETRATOR SEQUENTIAL (I.D.)	E	F
PENETRATOR SEQUENTIAL (C.C.D.)	G	H
DEFENDER SEQUENTIAL (I.D.)	I	////////
DOUBLE SEQUENTIAL (I.D.)	J	////////

where I.D. denotes Individual Detection, C.C.D. denotes Coordinated Central Detection, and A through J denote possibly different attrition equations which correspond to the basic sets of assumptions.[1] For comparison purposes, we will briefly describe these equations.

SIMULTANEOUS (Individual Detection): The many-on-one case (B in the taxonomy) is the binomial attrition model discussed in KARR (1974). Using the notation given in Section 1 (with $k = p_2 + p_4$, and with D_0 and A assumed to be deterministic), the binominal attrition equation for defenders killing penetrators gives that $E[Y_A]$, the expected number of the A penetrators that are killed, is

$$(9) \qquad E[Y_A] = A(1-[1-\frac{k}{A}(1-[1-d]^A)]^{D_0}) \ .$$

The reader is referred to KARR (1974) for details and to ANDERSON, BRACKEN, and SCHWARTZ (1975) for a method to compute the expected number of defenders that are killed. Assuming one-on-one combat

---

[1] This taxonomy is meant to be illustrative but not exhaustive; many other sets of assumptions are possible.

(Case A above) reduces the expected number of penetrators killed to

$$(10) \qquad E[Y_A] = Ak(1-[1-\tfrac{1}{A}(1-[1-d]^A)]^{D_0}) \ ,$$

To see that (10) follows from (9), note that if k is replaced by 1 in (9) then the right side is the expected number of penetrators that are detected by one or more defenders. Assuming that only one of the defenders can engage each detected penetrator gives (10).

SIMULTANEOUS (Coordinated Central Detection): If the defenders can be coordinated and assigned to engage any particular detected penetrator, then it is reasonable to assume that one defender would be assigned to engage each detected penetrator, provided there are enough defenders available in the one-on-one case (C above), and that the defenders would be evenly proportioned among the detected penetrators in the many-on-one case (D above). For the one-on-one case we have

$$(11) \qquad E[Y_A] = k \sum_{j=0}^{A} \binom{A}{j} d^j (1-d)^{A-j} \ \min\{j;D_0\}.$$

For the many-on-one case,

$$(12) \qquad E[Y_A] = \sum_{j=0}^{A} \binom{A}{j} d^j (1-d)^{A-j} \{ (D_0 - [\![\tfrac{D_0}{j}]\!] j)(1-(1-k)^{[\![\tfrac{D_0}{j}]\!]+1})$$
$$+ (([\![\tfrac{D_0}{j}]\!]+1)j - D_0)(1-(1-k)^{[\![\tfrac{D_0}{j}]\!]}) \},$$

where $[\![\tfrac{D_0}{j}]\!]$ denotes the largest integer less than or equal to

$\tfrac{D_0}{j}$ (so $[\![\tfrac{D_0}{j}]\!] \le \tfrac{D_0}{j} < [\![\tfrac{D_0}{j}]\!] + 1$). An additional assumption behind

(12) is that multiple engagements against the same penetrator occur independently, so the probability that a penetrator engaged by n defenders is killed is given by $1 - (1-k)^n$.

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PENETRATOR SEQUENTIAL (Independent Detection): The one-on-one case (E above) is precisely the model assumed in Section 1 of this paper. The many-on-one case (F above) requires additional research and will not be discussed in this paper.

PENETRATOR SEQUENTIAL (Coordinated Central Detection): There There is no essential difference between Simultaneous Combat with Coordinated Central Detection and Penetrator Sequential Combat with Coordinated Central Detection in the one-on-one case, so (11) also holds for case G above. The many-on-one case (H above) is different from Simultaneous Combat because, for example, the Central Detector must decide how many defenders to allocate against the first detected penetrator without knowing either how many previous penetrators went through undetected or how many more penetrators will be detected. Suppose that D_0/dA is an integer. Then one reasonable way to make this allocation (if the total number of penetrators is known in advance to the central detector and if D_0/dA is an integer) is to allocate D_0/dA defenders to engage each of the first $[dA]$ penetrators that are detected, and to allocate the remaining defenders against the next detected penetrator.[1] This allocation gives

$$(13) \qquad E[Y_A] = \sum_{j=0}^{A} \binom{A}{j} d^j (1-d)^{A-j} \min\{j, [dA]\} (1-(1-k)^{D_0/dA})$$
$$+ \sum_{\ell=[dA]+1}^{A} \binom{A}{\ell} d^\ell (1-d)^{A-\ell} (1-(1-k)^{D_0(1-[dA]/dA)}) \; ,$$

provided that D_0/dA is an integer. (Obtaining a formula for $E[Y_A]$ when D_0/dA is not an integer requires only sufficient interest in the result.)

---

[1]Here $[dA]$ denotes the largest integer less than or equal to dA. Since dA is the expected number of penetrators that are detected, this alloca-tion is a proportional allocation of defenders according to the expected number of detected penetrators.

DEFENDER SEQUENTIAL (Independent Detection) and DOUBLE
SEQUENTIAL (Independent Detection): If defenders are encoun-
tered sequentially--whether by one penetrator at a time or by
all penetrators simultaneously--then only the wording of Assump-
tion 1 needs to be changed. The independence assumption con-
tained in Assumption 2 gives that all results of Section 1
apply directly to these two cases (I and J above). Indeed,
Assumption 2 is more plausibly fulfilled if defenders are
encountered sequentially.

In summary, the attrition process presented in Section 1
applies to cases E, I, and J in the taxonomy presented above.

c. Interpretation of our Assumptions

We will now discuss some physical situations in which the
assumptions of our attrition model, as presented in Section 1,
are satisfied to some extent.

Air-to-air combat motivated this model, and in certain cases
thereof the assumptions seem reasonable. Consider the following
physical situation: There are targets (cities, airbases, etc.)
and the defenders erect a barrier by flying patrols in a fixed
airspace between the targets and the homeland of the penetrators.
For simplicity we picture this airspace to be a rectangular box
(see Figure 1). One by one, the penetrators attempt to penetrate
through the barrier in order to attack the targets.
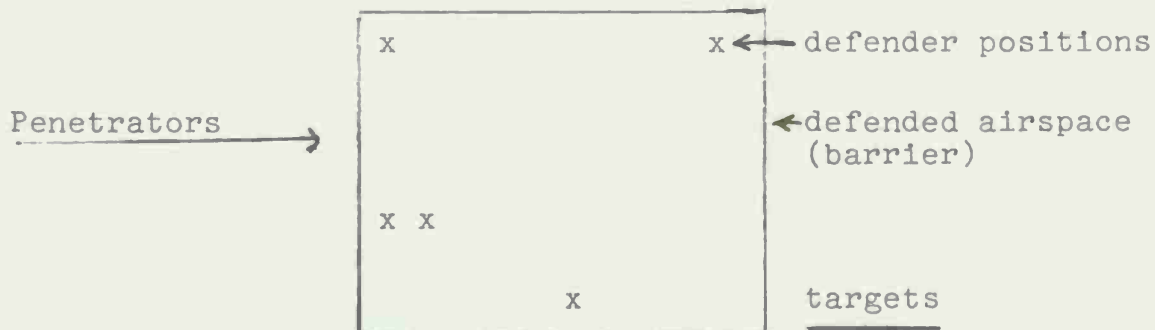

Figure 1

Assumption 1 will be discussed following the discussions of
Assumptions 2 through 5 below.

Assumption 2 appears to be hardest to satisfy. If one iden-
tifies the defenders by their positions in the barrier and in
addition one knows the spot at which a penetrator will attempt
penetration, then it isn't reasonable that all defenders should
have the same probability of making a detection; those that are
closest to the spot of attempted penetration certainly should
have higher probability. Without further clarification, this
argument fails to contradict Assumption 2 because it involves
conditional probabilities: given the locations of all defender
aircraft and the point of the attempted penetration, those
nearest to that spot will (under any reasonable assumptions con-
cerning the physics of detection) have higher probability of
making a detection. But in terms of absolute probabilities
the argument can be erroneous, for it requires information not
used for computing unconditional probabilities. First, suppose
that defenders are exogenously identified (for example, by tail
number) rather than by a property internal to the process, such
as position in the barrier. If we assume that defenders are
assigned to barrier positions in such a way that all assignments
are equally likely, then, even if the conditional probability of
a defender's detecting a penetrator, given all positional infor-
mation, depends on their relative positions, all defenders have
the same unconditional probability of making a detection (more-
over, actual computation of the detection probability d would
presumably involve such a conditioning argument). Note that no
assumption was required above concerning the distribution of the
point of attempted penetration.

The independence part of Assumption 2 is the most difficult
assumption to satisfy. Indeed, suppose that the barrier and de-
tection physics are such that, for any positions of the defenders
and the attempted penetration, at most one defender can make a

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detection (for example, the barrier has narrow depth and defenders are evenly spaced across its front). Or suppose that the defenders' positions are fixed and the penetrators all penetrate at the same place so as to saturate a strip through the barrier. In either case the independence part of Assumption 2 simply fails. On the other hand, if each defender chooses a position in the barrier independent of the positions of all other defenders, then independent detections of a given penetrator will follow and consequently the independence part of Assumption 2 will be fulfilled. In order that all defenders have the same probability d of detecting a particular penetrator, the random positions of the defenders must be identically distributed. The situation could be envisioned as that in which a penetrator passing within some critical distance r of a defender is detected; the ratio of $\frac{4}{3} \pi r^3$ to the volume of the defended airspace might then be thought of as the detection probability d.

Suppose that the defenders' positions are chosen according to an arbitrary probability distribution over the defended airspace, and that once chosen these positions remain fixed throughout all attempted penetrations, except as defenders leave to engage in one-on-one interactions. In addition, assume that each penetrator passes through the defended airspace along a straight line parallel to the long sides of the box (see Figure 2).
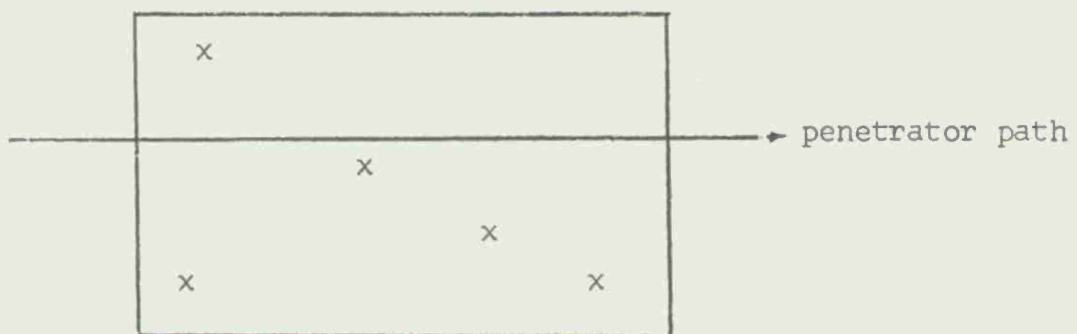

Figure 2

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Assume further that the entry point is uniformly distributed on the short side of the box and that different penetrators make independent choices of entry points, which are also independent of the positions of the defenders. Then:

a) Different defenders detect a given penetrator independently of one another, because of the independence of the defenders' positions;

b) If the probability that a defender detects a given penetrator is taken to be the fraction of the airspace within which his detection equipment can operate (that is, any penetrator passing within a certain distance of the defender is detected), then this probability of detection is the same for all defenders and all penetrators. (If penetrators did not make independent choices of paths this probability might not be constant; in particular if one penetrator passed through undetected and could relay this information back to other penetrators, then the succeeding penetrators would follow the same path, and be detected with probability zero.)

Hence in this case all parts of Assumption 2 are satisfied.

Alternatively, one could assume that each defender's position is uniformly distributed over the defended airspace and is independent of the positions of the other defenders, and that after each attempted penetration each remaining active defender chooses a new position, which is uniformly distributed, independent of the past history of the attrition process, and independent of the new positions chosen by the other remaining defenders. If, further, each penetrator chooses a path parallel to the long sides of the box independently of the paths chosen by other defenders but according to any probability distribution of the entry point on the short side (Figure 2 is still applicable) then again all parts of Assumption 2 are satisfied.

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Yet another situation in which independence holds is that of "running the gauntlet" (Defender Sequential or Double Sequential Combat, as described above) (see Figure 3).

```
┌─────────────────────────────────────────────────────────┐
│                                    Defenders             │
│                              x    x    x    x    x       │
│    path of penetrators ──────>                           │
│                                                          │
└─────────────────────────────────────────────────────────┘
```
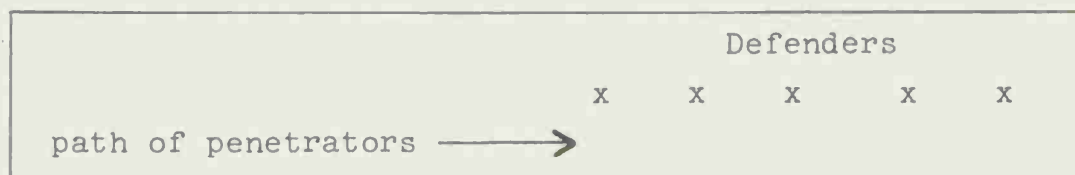
Figure 3

One thinks of this situation in the following terms: a penetrator (or, equivalently, all penetrators) is vulnerable to detection first by one defender, then by the second defender, ... , and so on to the last defender. Whenever a detection occurs, it results in an immediate engagement between the defender making the detection and the detected penetrator (or, in the Defender Sequential case, between the detecting defender and one of the penetrators he has detected).

An aspect of these situations is that they bring out in physical terms an interesting mathematical property of our model: a duality between defenders and penetrators. All probabilities involving number of engagements, computed in Section 1 above, are the same for m defenders and k penetrators as for k defenders and m penetrators, which can lead to computational simplicity. Other attrition distributions, of course, need not have this property (nor necessarily should they).

Assumption 3 principally concerns the timing of the interactions and the capabilities of the defenders, but it may also concern tactics as well. For example, given Assumption 5--that being engaged by one defender causes a penetrator to abort his primary mission--the tactic of the defenders may be to make as many one-on-one engagements as possible rather than to make fewer many-on-one engagements. Implicit in Assumption 3 is the idea that the defense is always able to engage a detected penetrator (i.e., the penetrator cannot evade engagement and

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continue to the target). This is not restrictive because the detection probability d can be assumed to represent only engageable penetrators.

Once Assumption 3 is made, Assumption 4 is both natural and general, so it will not be discussed further. The only point of possible disagreement is that the outcome is assumed to be independent of the history of the process; it may be that a defender is less effective the longer it has been on patrol. Allowing for such effects would add considerable mathematical complexity to the model.

The validity of Assumption 5 is related to the length of time over which the entire process of attempted penetrations is presumed to occur as well as to the inherent capability of the defenders. If the length of a representative engagement is long relative to the overall length of the battle, then it is plausible to assume that engaged defenders do not return to the barrier. A more symmetric argument applying to both defenders and penetrators is that an engagement exhausts the fuel or munitions (and, for penetrators only, causes the jettisoning of ordnance) to the extent that afterwards each will not attempt to resume its previous mission, but must return to its base. It may, however, participate in the next battle to occur. In Section 3 we consider some ways this assumption can be weakened.

As outlined above, the one-at-a-time attempts at penetration specified by Assumption 1 can be replaced by assuming that the defenders are encountered one-at-a-time. But the Markov nature of the process described in Section 1 requires sequential interaction in one sense or the other. If the tactic of the side attempting penetration is to send a fixed number of penetrators at a time, and the defender's tactic is to use a fixed number of defenders working in concert, then this case can be accommodated by replacing "penetrators" and "defenders" with

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"groups of penetrators" and "groups of defenders" in Assumptions
1, 2, 3, and 5, provided that those assumptions still hold for
groups. That is, if only Assumption 4 needs to be changed to
handle group-on-group interactions, then the model of Section 1
can be used, with Assumption 4 varied as appropriate. Assumption
5 appears to be the hardest to satisfy for groups, especially if
the number of penetrators in a penetrator group is greater than
the number of defenders in a defender group. But if Assumption 5
is satisfied, then Assumptions 2 and 3 are not implausible
(Assumption 1 only serves to define the process).

The applicability of this model to other forms of combat
requires further examination. In general all the assumptions
appear more realistic the smaller the scale (in time, space,
and number of combatants) of the process is assumed to be.
Large scale processes might possibly be treated by decomposi-
tion into independent engagements of smaller size. Of the
assumptions required, the hardest to satisfy would almost cer-
tainly be the independence part of Assumption 2, although one
can always use a model whose assumptions don't hold perfectly.
Indeed, this seems preferable to using a model for which no
underlying set of assumptions is known.

3.   GENERALIZATIONS

By adding various assumptions, one can extend the basic model of Section 1 to more general situations.

Perhaps the most desirable extension is to the case of heterogeneous forces of defenders and penetrators. To make such an extension requires additional assumptions, which we make here in the simplest form. Suppose first that there are J different types of penetrators. We need to specify the order in which the various penetrators make their attempted penetrations since, in general, the probability of a successful penetration will depend on the types (and not just the number) of penetrators which have previously attempted penetrations. Here is the minimal set of assumptions required:

6.   All parameters mentioned in Assumptions 1 - 5 are now functions of the penetrator type j.

7.   There are a fixed number k_j of penetrators of type j, for $j = 1, \ldots, J$. All penetrators of type j attempt their penetrations before any penetrator of type $j + 1$, for $j = 1, \ldots, J - 1$.

In other words, the penetrators attack in order by types (which requires the appropriate modifications to the taxonomy given in Section 2.b.). For each j denote by P_j the Markov matrix given by

$$
P_j = \quad
\begin{array}{c}
\\
0 \\
1 \\
2 \\
. \\
. \\
. \\
M
\end{array}
\begin{array}{cccccc}
0 & 1 & 2 & \ldots & M-1 & M \\
\left[\begin{array}{cccccc}
1 & & & & & \\
d(j) & 1-d(j) & & & & \\
0 & (1-(1-d(j))^2 & (1-d(j))^2 & & & \\
& , & . & . & & \\
& & . & . & . & \\
& & & . & . & \\
& & & & 1-(1-d(j))^M & (1-d(j))^M
\end{array}\right]
\end{array}
$$

where $d(j)$ is the detection probability corresponding to a type j penetrator. Corresponding to Theorem (2) we then have the following result.

(14) THEOREM. Under the extended set of Assumptions, the stochastic process D is a Markov chain, but does not have stationary transition probabilities. Instead, for $\ell \leq k_1 + \ldots + k_J$, we have

$$
P\{D_\ell = n \mid D_0 = m\} = \begin{cases}
P_1^\ell(m,n) & \text{if } \ell \leq k_1 \\
\left(P_1^{k_1} P_2^{k_2} \cdots P_{j+1}^{\ell-(k_1+\ldots+k_j)}\right)(m,n) & \text{if } k_1+\ldots+k_j < \ell \\
& \quad \leq k_1+\ldots+k_{j+1}
\end{cases}
$$

While the computations may be quite tedious, the concepts involved are simple. Given these probabilities, all other computations follow as in Section 1. In general, the number of defenders initially facing a given type of penetrator will be a random variable. But the apparatus of Section 1 is quite capable of handling this. For random numbers of the different types of penetrators, Theorem (14) fails; some computations are still possible, although quite complicated.

For different types of defenders one would, analogously, assume the presence of several barriers (of whatever physical

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form is felt to be appropriate) through which penetrators would pass successively. (Here random numbers of different types of defenders can be handled.) The mathematics of Section 1 has the necessary generality for a fully hetero-geneous model, but the computational and bookkeeping problems could be formidable.

One can also include the possibility that a defender involved in, but surviving, an engagement, might return to the barrier. Suppose (along with Assumptions 1-5) we make the additional assumption:

8. There is probability q that a defender surviving an engagement returns to the barrier. If he does so, it is before any additional penetrations are attempted and whether he does is independent of past history of the process.

Then all that changes is the transition matrix of the (D_k) process, which now becomes the matrix P* given by

$$P^*(0,0) = 1$$

and, for $i \geq 1$,

$$P^*(i,j) = \begin{cases} (1-(1-d)^i)[(p_1+p_3) + (p_2+p_4)(1-q)] & \text{if } j = i - 1 \\ (1-d)^i + (1-(1-d)^i)(p_2+p_4)q & \text{if } j = i \\ 0 & \text{otherwise.} \end{cases}$$

Similar mechanisms can allow for the possibility that a penetrator surviving an engagement may continue to attack his target. In this case the transition matrix P remains unchanged, as do the results of the computations in Section 1, but now we are also interested in the stochastic process $(S_k')_{k \geq 0}$ defined by

S_k' = number of the first k penetrators which continue
    on to attack the target

   = $S_k + G_k$,

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where

G_k = number of first k penetrators which are detected
and engaged, not destroyed in the engagement, and
continue on to attack the target.

We make in this case the following additional assumption:

9. There is probability $\tilde{q}$ that a penetrator engaged but
not destroyed by the defenders continues on to attack the tar-
get. This occurs independently of the past history of the
combat process.

One then obtains the following characterization of the
stochastic process (S_k').

(15) THEOREM. Subject to Assumptions 1 through 5 and 9

$$P\{S_k' = m \mid D_0 = r\} = \sum_{\ell,j:\ \ell+k-j=m} P^k(r,r-j) \times$$

$$\sum_{n=\ell}^{j} \binom{n}{\ell} \tilde{q}^{\ell} (1-\tilde{q})^{n-\ell} \binom{j}{n} (p_2+p_4)^n (p_1+p_3)^{j-n}$$

for all $m \leq k$.

PROOF. We observe that, as a consequence of Assumption 9, for
$\ell \leq n \leq k$

$$P\{G_k = \ell \mid V_k = n\} = \binom{n}{\ell} \tilde{q}^{\ell} (1-\tilde{q})^{n-\ell} \ ;$$

we remind that V_k is the number of the first k penetrators which
are engaged but not destroyed. From Theorem (6),

$$P\{V_k = n \mid B_k = j\} = \binom{j}{n} (p_3+p_4)^n (p_1+p_2)^{j-n} \ ,$$

for $n \leq j \leq k$. Finally, it follows from (7) that

$$P\{S_k = k-j \mid B_k = j\} = 1 \ .$$

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Consequently, by independence,

$$P\{S_k = k-j, G_k = \ell \mid B_k = j\}$$

$$= P\{G_k = \ell \mid B_k = j\}$$

$$= \sum_n P\{G_k = \ell, V_k = n \mid B_k = j\}$$

$$= \sum_n P\{G_k = \ell \mid V_k = n\} \, P\{V_k = n \mid B_k = j\}$$

$$= \sum_{\ell \leq n \leq j} \binom{n}{\ell} \tilde{q}^\ell (1-\tilde{q})^{n-\ell} \binom{j}{n} (p_3+p_4)^n (p_1+p_2)^{j-n}$$

The result now follows by summing over j and ℓ and using the fact that

$$P\{B_k = j \mid D_0 = r\} = P^k(r, r-j) \; . \qquad \square$$

Calculations involving (S'_k) can now be made in the manner previously described.

Other possible generalizations of the model include an analogous continuous time process, penetrator arrivals at random times, engagements of positive duration, the possibility of attempted penetrations during duels, and so on.

4. COMPUTATIONAL ASPECTS

Implementation of the basic attrition model of Section 1 (or of the generalizations considered in Section 3) requires computation of powers of the appropriate transition matrix of the remaining defenders process (D_k). We consider this problem next, first in a general form and then specifically for the original matrix P of Theorem (2). The general argument is applicable to the extensions discussed in Section 3; for simplicity we state it in terms of the original matrix P given in Theorem (2).

P is a lower triangular matrix of the following form:

$$
P =
\begin{bmatrix}
P(0,0) & & & & & \\
P(1,0) & P(1,1) & & & & \\
 & P(2,1) & P(2,2) & & & \\
 & & & \cdot & \cdot & \\
 & & & & \cdot & \cdot \\
 & & & & & \cdot & \cdot \\
 & & & & & P(M,M-1) & P(M,M)
\end{bmatrix}
$$

We are interested in:

1. The matrix whose columns are the (right) eigenvectors of P--for the purposes of this section we denote this matrix by N,

2. The inverse of N, and

3. The product NDN^{-1} where D is a diagonal matrix with elements δ_i being the eigenvalues of P. (Note that P has M + 1 distinct eigenvalues if $0 < d < 1$.)

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Once these are available we have

$$P = NDN^{-1}$$

and consequently

$$P^k = ND^kN^{-1}$$

for each k. Since D is a diagonal matrix it follows that for each k

$$D^k = \text{diag } \{\delta_0^k,\ldots,\delta_M^k\}$$

and the computational problem is solved.

First, it is clear from the form of P that

$$\delta_i = P(i,i) = (1-d)^i$$

for each i.

If we put x_j to be the eigenvector of P corresponding to the eigenvalue δ_j, normalized so that $x_j(j) = 1$, then it follows that for $i \geq 1$ and each j

$$P(i,i-1)x_j(i-1) + P(i,i)x_j(i) = \delta_j x_j(i) \ ,$$

so that

$$x_j(i) = \begin{cases} \dfrac{P(i,i-1)}{\delta_j - P(i,i)} \ x_j(i-1) & \text{if } j \neq i \\ 1 & \text{if } j = i \ . \end{cases}$$

We note also that

$$x_j(0) = \begin{cases} 0 & \text{if } j \neq 0 \\ 1 & \text{if } j = 0 \ . \end{cases}$$

Hence by induction

$$
x_j(i) = \begin{cases} \dfrac{\displaystyle\prod_{k=j+1}^{i} P(k,k-1)}{\displaystyle\prod_{\ell=j+1}^{i} [P(j,j) - P(\ell,\ell)]} & \text{if } i > j \\[20pt] 1 & \text{if } i = j \\[10pt] 0 & \text{otherwise.} \end{cases}
$$

Since $N(i,j) = x_j(i)$ this computation yields the matrix N, which is seen to be lower triangular.

Next we consider the inverse of N. Since N is lower tri-angular and $NN^{-1} = I$, the identity matrix, it follows that for $j < i$

$$
x_j(i)N^{-1}(j,j) + x_{j+1}(i)N^{-1}(j+1,j) +\ldots+ x_i(i)N^{-1}(i,j) = 0
$$

and consequently

$$
N^{-1}(i,j) = \begin{cases} \dfrac{-\displaystyle\sum_{k=j}^{i-1} x_k(i)\, N^{-1}(k,j)}{x_i(i)} & \text{if } i > j \\[20pt] 1 & \text{if } i = j \\[10pt] 0 & \text{otherwise.} \end{cases}
$$

This completes the necessary computations.

For the specific matrix P of Theorem (2) (and presumably for matrices of generalized processes that have the same lower triangular form) we may use generating functions to compute the powers of P in closed form.

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(16) THEOREM. For each n, i, and j we have

$$P^n(i,j) = \begin{cases} \prod\limits_{\ell=j+1}^{i} (1-(1-d)^{\ell}) \sum\limits_{k=j}^{i} \dfrac{(1-d)^{kn}}{\prod\limits_{\substack{q=j \\ q\neq k}}^{i} ((1-d)^k - (1-d)^q)} & j < i \text{ and} \\ & i - j \leq n \\[2ex] (1-d)^{in} & j = i \\[2ex] 0 & \text{otherwise .} \end{cases}$$

Note that $P^n(i,j)$ is independent of M, the maximum possible number of defenders, except for the restriction that $i,j \leq M$.

Before giving the proof let us recall the definition of a generating function (or z-transform); we refer to JURY (1964) for further details. If f is a nonnegative function defined on the set $\{0,1,2,\dots\}$, then the generating function of f is the function $\tilde{f}$ given by

$$\tilde{f}(z) = \sum_{n=0}^{\infty} f(n)z^{-n} ,$$

defined for whatever values of z for which the series is absolutely convergent.

PROOF. Since, existence questions aside,

$$(I-z^{-1}P)^{-1} = \sum_{n=0}^{\infty} z^{-n}P^n ,$$

we have that

(17) $$z(zI-P)^{-1} = \sum_{n=0}^{\infty} z^{-n}P^n$$

and we will be done if we can invert the transform given in (17).

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According to Theorem (2) we see that $z(zI-P)$ is of the form

$$
A = \begin{bmatrix}
A(0,0) & & & \\
A(1,0) & A(1,1) & & \\
0 & A(2,1) & A(2,2) & \\
& & \cdot & \cdot & \cdot \\
& & & \cdot & \cdot & \cdot \\
& & & & \cdot & \cdot
\end{bmatrix} ;
$$

the precise values of the entries are momentarily irrelevant. Denoting by B the inverse of A we note that since A is lower triangular, B must be lower triangular, and

$$
\begin{array}{c}
A(0,0)B(0,0) = 1 \\
A(1,0)B(0,0) + A(1,1)B(1,0) = 0 \\
A(1,1)B(1,1) = 1 \\
\cdot \\
\cdot \\
\cdot
\end{array}
$$

More generally

$$
B(i,i) = \frac{1}{A(i,i)}
$$

for each i, while for $j < i$

$$
B(i,j) = - \frac{A(i,i-1)}{A(i,i)} B(i-1,j) ,
$$

which allows us to conclude on the basis of induction that for $j \leq i$,

$$
B(i,j) = (-1)^{i-j} \frac{\displaystyle\prod_{\ell=j+1}^{i} A(\ell,\ell-1)}{\displaystyle\prod_{k=j}^{i} A(k,k)} .
$$

But

$$
zI - P = \begin{bmatrix}
z - P(0,0) & & & & \\
- P(1,0) & z - P(1,1) & & & \\
0 & - P(2,1) & z - P(2,2) & & \\
& & & \cdot & \cdot \\
& & & & \cdot & \cdot \\
& & & & & \cdot & \cdot
\end{bmatrix}
$$

and therefore

$$
B(i,j) = z \frac{\displaystyle\prod_{\ell=j+1}^{i} P(\ell,\ell-1)}{\displaystyle\prod_{k=j}^{i} (z-P(k,k))}
$$

which we can expand in partial fractions as

$$
(18) \qquad B(i,j) = \prod_{\ell=j+1}^{i} P(\ell,\ell-1) \left[\sum_{k=j}^{i} \alpha(k) \frac{z}{z - P(k,k)} \right],
$$

where

$$
\alpha(k) = \left[\prod_{\substack{q=j \\ q \neq k}}^{i} (P(k,k) - P(q,q)) \right]^{-1}.
$$

The term in brackets in equation (18) can be inverted since it is the sum of easily invertible functions. Performing this inversion gives that

$$P^n(i,j) = \begin{cases} \prod_{\ell=j+1}^{i} P(\ell,\ell-1) \sum_{k=j}^{i} \dfrac{P^n(k,k)}{\prod_{\substack{q=j \\ q \neq k}}^{i} (P(k,k) - P(q,q))} & \begin{array}{l} j < i, \\ i-n \leq j \leq n \end{array} \\[2em] P^n(i,i) & j = i \\[1em] 0 & \text{otherwise.} \end{cases}$$

But

$$P(i,j) = \begin{cases} (1-d)^i & j = i \\ 1 - (1-d)^i & j = i - 1 \\ 0 & \text{otherwise,} \end{cases}$$

and hence we have $P^n(i,i) = (1-d)^{in}$ and, whenever $i - n \leq j < i$,

$$P^n(i,j) = \prod_{\ell=j+1}^{i} (1-(1-d)^\ell) \sum_{k=j}^{i} \frac{(1-d)^{kn}}{\prod_{\substack{q=j \\ q \neq k}}^{i} ((1-d)^k - (1-d)^q)}$$

which is the asserted result. $\qquad\qquad\square$

We would like to thank Joseph J. Bolmarcich of Daniel H. Wagner, Associates, for suggesting a simplification which led to the form of $P^n(i,j)$ given above.

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